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On Causal Dynamics Without Metrisation: Part I

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Synopsis

By a careful topological examination of certain aspects of measurement processes, one reaches the unorthodox conclusion that it is not legitimate to assume that three-space is metrisable throughout the universe. The examination consists of an identification of the role of the separation axioms in the conventional mathematical formulation of the information we extract from actual measurements. To provide a means of handling the limitations introduced by errors of measurement, a species of local structure is proposed that is a particular example of a Zeeman tolerance space. Some homology properties relating to the way in which measurements containing errors are adjoined are deduced for spaces possessing the proposed structure, and some simple properties of transformations of such spaces are given.

1. *Introduction*

1.1. *Résumé*

It is of interest to the mathematician to discover if one can formulate dynamics in terms that completely discard the notion of metrisation. The implication of such a proposal is that an algebraic approach must be sought, as opposed to a global differential geometric approach.

It is of interest to the physicist to seek formulations of dynamics and of the notion of causality that are free from metrisation, for, as Section 4 leads one to understand, it is not legitimate to assume that even ordinary three-space is metrisable throughout its entirety. There is also the hope that such an approach may help to throw some light upon difficulties now being encountered in the understanding of sub-nuclear and cosmological problems.

The first principal section of this paper investigates the roles of the separation axioms in physical spaces, an investigation that is carried out carefully in order to expose as many matters as possible that play a critical part in assessing the applicability of various kinds of topological spaces one may hope to be significant in physics. The outcome

† This work was undertaken and completed whilst the author was at the Post Office Research Station, Dollis Hill, London, N.W.2.

of the investigation is the discovery that one is left with much weaker topological properties than is usually assumed for spaces—nothing stronger than locally compact Hausdorffness may be assumed.

The second section provides a similar study of the topological properties of 'measurement processes' (defined in the first section), and 'dynamical processes', enabling one to infer a topology for the 'space of physical conditions' (a notion also coined in the first section).

The third section uses the deductions of the previous two sections in conjunction with Smirnov's general metrisation theorem, demonstrating that even if a space of measurements is locally metrisable, one may not assume that it is metrisable throughout its entirety.

It is found that the plain fact of finite, non-zero errors in measurements introduces some inherent difficulties into the discussion about topological properties of transformations between events that are labelled by the values of measurements. To avoid those difficulties, a species of local structure (in the sense of Ehresmann) is proposed. This structure, named a 'Planted Structure', incorporates the notion of error in such a way that errors are ignored; this is accomplished by constructing a space modulo error.

The fourth section contains firstly, the definition of the local structure, secondly a deduction of some of its basic topological properties, and finally presents some simple properties of transformations between such structures. The fifth section is concerned with some of the problems in section four that pertain to the homology of planted structures.

1.2. *Retrospect and Prospect*

There is only one book devoted to an essentially topological examination of dynamics (Gottschalk & Hedlund, 1954), a work based upon the hypothesis that dynamical transformations belong to a topological group.† Contrasted with that, the approach used here is much more physically intuitive. Here the purpose is to look very carefully at the tacit connotations and implications contained in the customary use of the term 'measurement process', and to find out exactly how to give expression to those connotations and implications by means of topological notions. This task does not appear to have been attempted before. That the effort is worthwhile becomes obvious, owing to the

† Since the time of writing this paper there has appeared a book on the modern approach to classical mechanics in global language. (See Abraham (1967).) That exposition is certainly coordinate-free, but not in the sense introduced by the merely relational approach to dynamics that is developed in the later parts of this work.

unexpected conclusions that are drawn, e.g. Section 4.4, RMK(20), PR(5), and Section 6.3, RMK(32).

This paper is essentially incomplete, because it suggests many further basic studies that may raise some insoluble problems. However, it will be seen in the second paper in this series ('On Causal Dynamics Without Metrisation: II') that there is a realisation of a dynamical process satisfying a metric-free condition for causality of dynamical processes, that consists of the familiar paraphernalia of Minkowski space-time, but donated with Zeeman's 'fine topology'.

1.3. *Notations*

In order that the presentation may be clearer, a system of abbreviations has been introduced to enable the reader to learn the importance attached to various statements. The abbreviations are as follows:

ANS( )	Answer	ASSN( )	Assumption
AX( )	Axiom	COR( )	Corollary
CVN( )	Convention	DF( )	Definition
EQN( )	Equation	EX( )	Example
HYP( )	Hypothesis	IFOF	If and only if
LMA( )	Lemma	PR( )	Proposition
QU( )	Question	RMK( )	Remark
TH( )	Theorem	]	Q.E.D.

A double bracketing, e.g. LMA((1066)), indicates an alternative, but seldom cited, form of the abbreviation tabulated by single brackets. Some double compositions of abbreviations are used as well, e.g. RMK/DF(1216), in which case the number always refers to the indexing set of the first abbreviation, which, therefore, is considered to denote the more appropriate interpretation of the labelled statement.

2. *The Separation Axioms in Physical Spaces*

2.1. *Motivation*

Probably the first physical space which comes to the mind of the reader is $(3 + 1)$-dimensional space-time, for it is used to describe the locations of objects and events directly involved in his everyday life. There are, of course, many others constructed on a more mathematical footing (e.g. momentum-energy space and angular momentum-spin space), but the more immediately apprehendable space-time of $(3 + 1)$ dimensions is of best use for the relatively simple discussion which follows.

A common feature of the way in which quantities or entities are measured is that different devices are used for different ranges of 'sizes'. This is a direct result of the different ways of obtaining accurate measurements (i.e. those methods of measurement with very small errors) which are forced upon one in the different cases to hand. For example, to measure the size of a sheet of letter paper one would use a foot rule, to measure its thickness a micrometer, whilst to measure the length of a roll of wallpaper one would use a very long, flexible, tape-measure. Similar contrasts can be made with time-pieces as the reader wishes. The critical point of argument is that whenever an extensive scale of measurement is brought into use in physics, the whole range of values can in general be compassed by *several* pieces of apparatus only: this is certainly true when standards are compared with devices being calibrated. Therefore, although the notion of 'length', for example, is one with which each one of us feels quite familiar when speaking of inter-atomic, macroscopic or interstellar distances, the familiarity and understanding we possess (which is so often thought of as almost intuitive), arises only from the existence of a number of very well related measuring techniques. It is this multiplicity of measurement processes which prompts the following DF(1 and 2), as well as something of the examination of the advance of the so-called 'Separation Axioms' in the study of physical spaces.

2.2. *Measurement Processes*

Without restricting ourselves to any particular example of a measurement, we shall proceed to give as general a definition of a measurement process as possible. To begin with we need a set of physical 'conditions' C.† If we consider C to contain all possible conditions which can be measured, and is thereby ascertained to possess 'values' of a particular physical property to which the measuring devices are sensitive, then we know that more than one device will have to be used. Consequently, each device will have a different 'domain' of applicability. We therefore must assume that C can be subdivided into a collection of subsets, $\{C_i\}$, which can be ordered into a sequence—by the subscript i—which corresponds to the ordering of the measuring devices into a sequence. There is, of course, a natural sequence. In most cases, it is given by the absolute magnitudes of the

† The word condition is used in preference to the more well defined and current notion of 'state', so that at a later stage one may be free to introduce a more precise notion which can be called a 'state'; moreover, the word 'condition' leaves us free to suppose that even though a particular state may be under observation (that is to say, the subject of the measuring process) there may also be some special circumstances under which the measuring apparatus is being used. For example, a very cold micrometer may be used to measure the diameter of a very hot rod, circumstances which clearly necessitate accounting for.

ranges of numerical values which the measuring devices produce as a result of their operation. What is more, if we are to be able to assume that, given the natural sequence of the C_i, measurements upon adjacent C_i's are related in a sufficient degree to allow comparison of measurements, then it is necessary to assume that adjacent C_i's are not disjoint.

The second and third of the three mathematical devices we need for this study of measurement processes, are a set of mappings corresponding to a set of measuring devices, and a parameter space in which values of the measurements are written. We may note that the measuring devices may be labelled by the same i's as the C_i's, and the subspaces of the parameter space in the same way too. Before we give any definition, there is one more point to consider—namely that a measurement of one particular property may not uniquely define a physical condition (or state). For example, the energy of an electron in a hydrogen atom may have a fixed value for several different values of the n, l, m, j, quantum numbers. If we can assert with certainty, for the purposes of a particular class of phenomena, that there is a unique physical condition corresponding to each measured value of a certain property, it is natural to call the measurement (process) a 'simple' one. The definition of a measurement process may now be given.

NTN(1): J_m is the set of all integers modulo m. In particular J_∞ is the set of all integers, and $J_\infty^{(+)}$, $J_\infty^{(-)}$ the sets of all positive and negative integers respectively, taking $0 \in J_\infty^{(+)}$, i.e. $J_\infty^{(+)} \cap J_\infty^{(-)} = \phi$, where ϕ is the empty set.

DF(1): Let $I \subset J_\infty$ be discrete and have finite cardinality. Given a set of measuring devices $P = \{P_i\}$, $i \in I$, a set of physical conditions C possessing a collection of subsets $\{C_i\}$, $i \in I$, and a space of parameters F, then to each particular measuring device P_i there is associated a mapping p_i called a *measurement sub-process* whenever the following conditions are satisfied:

$$\left.\begin{array}{ll} CDN(1): & p_i(C_i) \subset F_i \subset F \\ CDN(1): & (\bar{C}_i \cap C_{i\pm1}) \cup (C_i \cap \bar{C}_{i\pm1}) \neq \phi \\ CDN(3): & \bigcup_{i \in I} C_i = C \end{array}\right\} \begin{array}{l}\text{Condition expressing}\\ \text{the possibility of}\\ \text{measurement}\\ \text{comparisons, and in}\\ \text{which } \bar{C}_j \text{ denotes the}\\ \text{closure of } C_j\end{array}$$

DF(2): The mapping $p = \bigcup_{i \in I} p_i$ induced by P is called a *measurement process*.

RMK(1): Therefore a measurement process is basically a pre-sheaf over C with values in the elements of $\mathscr{F} = \{F_i i \in I\}$.

Before defining simplicity for processes [see Section 3.2, DF(10)], we need to examine the topological implications of errors in measurement. It cannot be circumvented by any means that errors are part and parcel of our perception, although they may be so small as to produce only an insignificant modification of the relationships we perceive as a result of our measurement operations, therefore any serious topological study of any notions in physics must take them into account. It will now be shown that the presence of errors directly brings to us consideration of the so-called Separation Axioms.

2.3. *The Notion of Topological Spaces*

The actual intent of a measurement—at least as far as the theorist is concerned—is to obtain a precise number which may be substituted into an equation, so that, by whatever particular condition of appraisal the equation may represent, either a judgement upon the validity of a theory may be made or else some other useful quantity may be calculated. Although the intent is to obtain a *precise* value from the measurement, we know well enough that in actual practice it is impossible to obtain complete elimination of error from *any* form of measurement. It certainly may be possible to reduce the fractional error in a measurement to such a low level that for all practical purposes it may be discounted and ignored, but from the topological aspect this is not true. The reason for this latter statement becomes more apparent by considering the definitions for a topological space and a limit point.

Let us suppose that in the space of parameters F, the point $a \in F$ represents the actual value of a quantity being measured. The measuring device—supposing that it is accurate—will register a value very close to a, but because of its inherent error the actual value recorded, a' say, will fall within certain limits about a; let us denote the collection of all possible a' by $O(a)$. Of course it is always possible that the precise value a will in actual fact be registered by the measuring device—i.e. $a \in O(a)$—but it is quite plain that if it is known that the measuring device has a certain inherent error, then it is impossible to make any precise statement about the actual value of the property (being measured) of the physical condition. So the most which can be said, is that the measuring device will register a value which lies within a certain neighbourhood of the 'true' value; that is to say the 'true' value is that which would be registered if the measuring device registered completely unambiguous values.

Now, although the last sentence can be considered to convey the truth, it does so, as it were, by using false principles. The 'real' truth

—to play with words—is that the value of the property of the physical condition lies within a neighbourhood, contained in F, 'centred' on the value registered by the measuring device; that is to say one should consider the expression $a \in O(a')$ rather than $a' \in O(a)$. The reason is simple to understand: it is that one can never assert† the result of a particular measurement; knowledge gained from measurement operations is always essentially *a posteriori* in nature. (This is, of course, in the sense that if one uses a measuring device with an error characterised by ε, then one cannot say that it will produce a registered value with error less than ε.) A further untruth contained in the sentence in question is the implication that the value of the property (being measured) of the physical condition belongs to the condition as some kind of Esau's blessing—in fact anyone who in recent times disputed such an assumption was treated like a potential Jacob—whereas, the value is no more than a convenient label assigned to the condition, in order that certain invariant relationships between the condition itself and other conditions (with different values of the property) may be classified in a manner acceptable to our senses of perception.‡

It is clear from this discussion that the most fundamental entities that are useful from the topological viewpoint, are the neighbourhoods lying in the space F of parameters; there neighbourhoods are inferred to exist by virtue of the activity of making measurements. It is true that one can always assume the existence of points in F, but they must be considered to exist only in the sense of some limiting process: namely, that if F is covered by a set of neighbourhoods of 'diameter' ε —where ε is a measure of the error of the measurement process—then points are yielded by the measurement process in the limit $\varepsilon \to 0$. This limit corresponds to the transition from the pre-sheaf structure to the limiting sheaf structure of the measurement process.

RMK(2): In practice it is impossible to reduce errors in a measuring process to zero, therefore one must always say that a measuring process maps a physical condition into a *neighbourhood* of the space of parameters used to characterise the measurements' values.

RMK(3): It is also quite clear that if the measuring process is carried out upon every possible different physical condition, then a set of

† The word 'assert' is used here not in the sense of 'predict', but in the sense of making an *a priori* true statement.

‡ If the reader is unfamiliar with Professor David Bohm's masterly analysis of perception and its relation to the physical sciences, he would be well advised to acquaint himself with it. See the appendix to *The Special Theory of Relativity* (Benjamin, 1965).

neighbourhoods, say $\mathcal{O} = \{O_i\}$, will be generated which will cover that part of F (the parameter space of measurement values) which is accessible by means of that particular measurement process; let us denote this subset of F by F'. In terms of the symbols introduced in DF(1 and 2) this becomes:

$$p(C) \subset \cup_i O_i \equiv F' \subseteq F$$

RMK(4): Notice, furthermore, that the set of neighbourhoods, $\mathcal{O} = \{O_i\}$, must be denumerable by reason of the cardinality of each O_i being greater than unity, which is to say: (i) no neighbourhood can consist of a single, precisely valued point; (ii) in order for two measurement values to be unambiguous, the neighbourhoods they generate must be disjoint. The consequence of these two requirements is that a (discrete) tabulation of distinct measurement values is brought about whenever an extended set of measurements is made by either an individual, or produced by a continuously operating, automatic 'monitoring' device. In the former case, $\mathcal{O}$ is countable, *de rigeur*—unless one counts a schizophrenic physicist as a non-rational person —and in the second case by reason of a modulo non-zero-error argument.

Having now introduced some sort of covering for F, we must go a little further and examine the physical significance of the notion of a limit point before any (mathematically) sensible discussion of the notion of a topological space can be entered upon.

2.4. *Limit Points*

Now it is clear that if one wishes to be certain that a measured value relating to a particular physical condition is a value with some significance, the measurement process will be applied several times to the condition (which we must assume to be prepared each time so that is is specified by some criteria). In the case of statistically controlled responses to the measurement process, a very large number of measurement values indeed must be obtained, in order to establish that a certain statistical pattern of values is repeatedly obtained. Only under these strictures can one begin confidently to assert that a definite (i.e. particular) physical condition is being recorded by means of the measuring process. This necessity for repeated confirmation is an essential ingredient of the nature of *Physics*, considering Physics as a formal study pursued by a collection of people: these matters are considered in the second paper in the series, Section 2.5, ASSN(10), where it is stressed that repeated confirmation of an experience, by any participant in Physics, is the most important characteristic of the study one can find. Consequently, one must consider any physical condition as being associated with a number of (the minimal) neighbourhoods in the parameter space of measurement values, if one wishes to express the postulate that Physics is essentially a corporate study. It is just this natural need for a multiplicity of such neighbourhoods which offers a natural way of discussing limit points, and we may do so in the following way.

Suppose that a particular physical condition c (specifiable by a set of criteria which we do not need to know for the purposes of this discussion), which, mapped by a measuring process p into the parameter space F, gives rise to a set of measurement values $\{a_i\}$ together with the set of corresponding neighbourhoods in F, denoted by $\{O(a_i)\}$. In symbols this is:

$$p_\varepsilon(c) \subset O_\varepsilon(a_i^{(\varepsilon)}) \qquad (i = 1, 2, \ldots, n)$$

where n is equal to the number of measuring operations performed, and where the super-/sub-script ε has been temporarily introduced in order to indicate that p_ε and O_ε are associated with an error, inherent in the measuring device, characterised by the quantity ε. If in the limit $\varepsilon \to 0$ we have the value a, then we may write:

$$a = \lim_{\varepsilon\to 0} p_\varepsilon(c) = \lim_{\varepsilon\to 0} O(a_{\varepsilon i}^{(\varepsilon)}) \Rightarrow \lim_{\varepsilon\to 0} a_i^{(\varepsilon)} = a$$

$$\Rightarrow a \in \cap_i O_\varepsilon(a_i^{(\varepsilon)}) \Rightarrow \lim_{\varepsilon\to 0} \cap_i O_\varepsilon(a_i^{(\varepsilon)}) = a$$

These relations show that a is a limit point of F, since every neighbourhood defined by the measurement process contains at least one other point; typically, one of the set $\{a_i\}$: notice that here we are assuming that $a = \lim_{\varepsilon\to 0} p_\varepsilon(c)$ exists. There is also another means of indicating that a precise value $a = \lim_{\varepsilon\to 0} p_\varepsilon(c)$ may be considered as a limit point of F: for if one considers a sequence of progressively more accurate measurements as defining a filtered set of neighbourhoods, that is to say $a \in O_{\varepsilon_0}(a_0) \subset O_{\varepsilon_1}(a_1) \subset O_{\varepsilon_2}(a_2) \subset \ldots \subset O_{\varepsilon_m}(a_m)$, where $\varepsilon_i < \varepsilon_{i+1}$, and where $m + 1$ is the number of measurements made, then each set $O_{\varepsilon_i}(a_i) - O_{\varepsilon_{i-1}}(a_{i-1})$ is both contained in $O_{\varepsilon_i}(a_i)$ and contains a point different from $a = \lim_{\varepsilon\to 0} p_\varepsilon(c)$; also, the neighbourhood $O_{\varepsilon_0}(a_0)$ contains both a and a_0. We may therefore reasonably assume, on the basis of these arguments, that a measuring process does not give rise to limit points in the space F, but does give rise to a set of neighbourhoods adequate for describing the topology of the subsets of F accessible by measurement.

2.5. *Closed and Open Sets*

DF/NTN(3): If X is a subset of a set S, which has been given a topology, then $\bar{X}$, the *closure* of X, is the union of the set X and set of all limit points of X.

In order to use the notion of *closed* sets, the definition above makes it clear that one must be able to specify *all* the limit points of a given set. Now, whilst an actual measurement does specify a set, namely a neighbourhood lying in F, and infers a limit point of the measurement process, namely lim $p_\varepsilon(c)$, it is not *a priori* possible to assert that any measurement process can detect every possible limit point (representing a measurement value in the space of parameters F) for the very simple reason that each process has a minimum error which cannot *a priori* be said *not* to mask any effects of finer detail. This last remark is not meant to preclude the case where a very successful theory combined with experiments of sufficient accuracy does in actual fact indicate that it may be possible to know all limit points of a given set; for example, physicists probably accept as sufficiently well demonstrated that quantum mechanics describes the spectrum of hydrogen, so that they may take the limit point of the set of bound energy levels as characterised by the state with zero total energy. However, if one wishes to construct a mathematical theory for physics which can always be enriched in structure without changing its basic form, then one must start from the weakest assumptions—and here as we have seen, it is necessary to assume that:

ASSN(1): In general, a measurement process defines open sets of the value parameter space, and closed sets may be specified by hypothesis only.

Here the well-known reciprocity between closed and open sets has been used, namely that if a set is not closed then it is open. Having arrived at the notion of an open set in relation to a measurement, we find ourselves at the doorstep of the Separation Axioms.

2.6. *Separation Axioms*

$AX(T_0)$: Given two points of a topological space S, at least one of them is contained in an open set not containing the other.

$AX(T_1)$: Given two points of S, each of them lies in an open set not containing the other.

$AX(T_2)$: Hausdorff Axiom: Given two points of S, there exists *disjoint* open sets, each containing just one of the two points.

Again it is necessary to consider the result of measuring processes in order to discover the separation properties—according to the

Separation Axioms—of the parameter space of measurement values, the latter being the basic space available for constructing theories of physical phenomena. All that one needs to notice is that one must be able to state that the (respective) minimal neighbourhoods, in F, of two measured values do not overlap (i.e. they are disjoint), in order to be able to assert that the two measured values are distinct (i.e. that they are clearly distinguishable from each other); for otherwise it would not be possible to assert that the two actual values (in the limiting sense of $\lim_{\varepsilon\to 0} p_\varepsilon$) are distinct, the simple reason being that they would then both lie in a subset of the parameter space of values which could be strictly contained within a minimal neighbourhood. As we have already found that the minimal neighbourhoods must be considered to be open, the remarks above lead us to the conclusion that the topology defined on the parameter space of values, F, by means of a measurement process, satisfies the Hausdorff Axiom at least.

For convenience, let us suppose that a, $b \in F$ are two distinct measurement values; in symbols this is written $O_{\varepsilon_0}(a) \cap O_{\varepsilon_0}(b) = \phi$. In particular, this relation expresses the Hausdorffness of F. It is quite simple to see in a satisfactory intuitive way—but not too satisfactory a mathematical way†—that the topology of F gained from a measurement process does have the T_0 and T_1 properties. For, provided ε_0 is the minimal error, it is always possible provided the condition $O_{\varepsilon_0}(a) \cap O_{\varepsilon_0}(b) = \phi$ holds to find an $\varepsilon_2 > \varepsilon_0$ such that there holds $a \in O_{\varepsilon_2}(b)$, which demonstrates that T_0 is satisfied. Furthermore one can also choose an $\varepsilon_1 > \varepsilon_0$ such that $O_{\varepsilon_1}(a) \not\subset O_{\varepsilon_1}(a) \cap O_{\varepsilon_2}(b) \neq \phi$ and $O_{\varepsilon_2}(b) \not\subset O_{\varepsilon_1}(a) \cap O_{\varepsilon_2}(b) \neq \phi$, which clearly shows that T_1 is satisfied.

There are two more commonly used separation axioms, which we must not omit to consider if metrisation of physical spaces is to be seriously studied. They are as follows:

AX(T_3): If X is a closed set in a topological space S, and $y \in S$ is a point such that $y \notin X$, then there exist disjoint open sets U, $V \subset S$ such that $X \subset U$ and $y \in V$.

DF(4): A T_3-space which is also a T_1-space is called a *regular space*.

† The difficulty is that whilst the separation axioms speak of points as if they actually are well-defined, practicable measurement processes define open sets only. 'Physical' points are assumed to exist only in so far as the limiting process $\lim p_\varepsilon(c)$ is assumed to exist. Any such assumption can never be ultimately tested because of our inability to build completely error-free measuring devices—to the strict mathematician, then, we are actually begging the question owing to this particular reason.

TH(1): A T_1-space S is regular IFOF for each point $x \in S$, and for each open set $U \subset S$ with $x \in U$, there exists an open set $V \subset S$ such that $x \in \overline{V}$ and $V \subset U$.] Hocking & Young (1960), TH(2.4).

AX(T_4): If X, $Y \subset S$ are disjoint closed sets, then there exist disjoint open sets U, $V \subset S$ such that $X \subset U$ and $Y \subset V$.

DF(5): A T_1-space which is also a T_4-space is called a *normal space.*

An immediate difficulty is presented by $AX(\mathrm{T}_3)$ in that a closed set has to be specified, and, as has been stated in ASSN(1), it is not possible, in general, to specify a closed set by means of a measurement process. We may remark that if a closed set X were postulated, then $\mathrm{AX}(\mathrm{T}_3)$ would demand that it be possible to cover it completely by a union of open neighbourhoods defined by a set of measurements that can actually be performed. But even an application of TH(1) is inconclusive, because it is only possible to assert *a priori* that all the limit points of a subset of the parameter space of measurement values belong to the subset, whilst it is *never* possible *in practice* to make even an error-free *a posteriori* statement about any particular measurement value. Therefore we may say:

ASSN(2): In the completely general case it is impossible to assert regularity *for a parameter space of values. However, when a measurement process appears to yield well-defined limit points for sets of measurements, it may be a useful approximation to assume that the parameter space be regular.*

RMK(5): It is quite clear that if a measuring process p_{ε_0} with minimal error ε_0 be assumed to give the parameter space F of measured values a regular topology wherever $p_{\epsilon_0}(c)$ has a meaning, then if $X \subset F$ be assumed to be closed, then for any $y \in F$ there must hold $O_{\varepsilon_0}(y) \cap X = \phi$, otherwise it would not be possible to assert that y be not a limit point of X [see TH(1)].

RMK(6): A similar assumption to ASSN(2) must be stated about the *normality* of the parameter space of measurement values.

2.7. *Compactness*

Before we finally reach a conclusion about the nature of the topology of the parameter space of measurement values, it is necessary to examine the 'physical' interpretation of the notions of compactness and paracompactness.† First of all some fundamentals must be given.

TH(2): $X \subset S$ is compact IFOF every open covering of S contains a finite covering of X.] Hocking & Young (1960), LMA(1.20).

† For paracompactness see Section 4.2.

TH(3): Let S be a Hausdorff space and $X \subset S$ be a compact subset, and let $y \in S$ and $y \notin X$. Then there exist disjoint open sets U, V such that $y \in U$, $X \subset V$.] Hocking & Young (1960), TH(2.1).

COR(1): Compact sets in a Hausdorff space are closed.] Hocking & Young (1960), COR(2.2).

DF(6): A space S is said to have the *finite intersection property* provided that if $\{A_i\}$ is any set of closed subsets such that any finite number has a non-empty intersection, then the total intersection $\bigcap_i A_i$ is non-empty.

TH(4): Compactness is equivalent to the finite intersection property.] Hocking & Young (1960). TH(1.22).

RMK(7): Owing to the finite, non-zero size of the neighbourhoods defined in the parameter space of values F by a measurement process, any subset of F accessible by the measurement process can be covered by a finite number of neighbourhoods. This allows us to state:

ASSN(3): The topology endowed by a measurement process upon a parameter space of measurement values is locally compact *and* Hausdorff.

Here the term 'locally compact' has been introduced without definition because it has a clear intuitive meaning in this context. It may be stated more precisely as follows:

DF(7): A topological space S with an open covering $\{O_i\}$ is said to be *locally compact* if an open neighbourhood $U \subset S$, $S - U \neq \phi$, of an arbitrary point $x \in S$ can be covered by a finite number of the O_i.

It is not very difficult to prove the following proposition:

PR(1): All compact spaces are locally compact, but not all locally compact spaces are compact.]

RMK(8): From ASSN(3) and TH(2) it is plain that a subset of F obtained by measurements is compact IFOF the subset can be experimentally detected—for this involves a finite number of defining neighbourhoods only, otherwise one would have to live an infinite time in order to carry out a non-finite number of experiments giving different neighbourhoods.

The COR(1) of TH(3) introduces an unexpected result. It seems to contradict the conclusion stated in ASSN(2), which was arrived at from the impossibility of enumerating all possible limit points of a measurement (sub-) process. ASSN(3) and TH(3) together infer the existence of closed sets in the parameter space F, since F is already recognised as being Hausdorff at least. However, a measurement process does not define a topology over the whole of F, but over a subset only [see RMK(3)]. Consequently one can only interpret the combination of ASSN(3)TH(3) as meaning that closed sets exist in F in a local sense: that is to say, wherever a measurement process has not been used actually to verify that a compact topology may be given to a region of F, it is not possible—excepting by hypothesis alone—to assert that F is compact, and so contains definable closed sets (in terms of the measurement process's neighbourhoods as just explained), which would, in turn, allow one to assert F to be regular.

TH(5): Every compact Hausdorff space is normal.] Hocking & Young (1960), TH(2.3).

RMK(9): Since a parameter space of measurement values is only locally Hausdorff, we may not assert that its topology endowed by a measurement process is normal—although it may be—but we may in an obvious sense say that it is locally normal.

RMK(10): Also it may only be asserted that a parameter space of measurement values has the finite intersection property in a local sense.

RMK(11): It will now be shown that the definition of local compactness of a space given in DF(7) in a physically interpretable way, is equivalent to the customary definition—for example see Hocking & Young (1960), Section 2.10—if one includes the Hausdorff condition as in ASSN(3), and does not include regions near the boundary of a locally compact subset of a parameter space of measurement values. First of all, let us state the necessary definitions:

DF(8): A space S is said to be *locally compact at a point* $x \in S$ if there exists some open set $U \subset S$ with $x \in U$, and such that $\bar{U}$ is compact.

DF(9): A space is said to be *locally compact* if it is locally compact at every point.

It may immediately be noted that DF(9) implies DF(7), for if $\bar{U}$ is compact then so is U. To show that DF(7) implies DF(9)—at least as far as physics is concerned (and assuming that the considerations of this paper are relevant to physics)—we need to lean on the local Hausdorffness of parameter spaces of measurement values. Local compactness, in the sense of DF(7), combined with local Hausdorffness gives rise to local normality—by TH(5) interpreted locally. This condition means that closed sets exist locally;† denote such a closed set by $\bar{X}$. Then $X \subset \bar{X}$, and $\bar{X}$ is compact by reason of local compactness in the sense of DF(7). Therefore under the condition of ASSN(3),

† As well as disjoint closed sets coverable by disjoint open sets.

we have formally shown that DF(7) implies DF(9). However, this proof is not universally valid in the parameter space of measurement values, F, even where F bears some topology endowed by the measurement process, owing to the finite, non-zero size of the minimal covering neighbourhoods defined by the measurement process. All that can be said is that $O_{\varepsilon_0}(a_0)$ is the smallest set in the vicinity of a precise measurement value; so that in order to satisfy the local compactness condition of DF(8, 9), its closure must lie within a compact set. As we have already assumed that the vicinity of a particular precise value may be considered compact—for this is what the term 'locally compact' means in ASSN(3)—then a sufficiently large number of measurement neighbourhoods $O_{\varepsilon_i}(a_i)$, $\varepsilon_i > \varepsilon_0$, in the vicinity of $O_{\varepsilon_0}(a_0)$ and intersecting $O_{\varepsilon_0}(a_0)$, will cover $\overline{O_{\varepsilon_0}(a_0)}$ compactly. If $O_{\varepsilon_0}(a_0)$ were supposed to be very close to the edge of a larger compact region of F, it is not necessarily true that the 'gap' between $O_{\varepsilon_0}(a_0)$ and the edge of the region would contain, or would be as 'wide' as an ε_0-neighbourhood.† In such a case, the neighbourhood $O_{\varepsilon_0}(a_0)$ could not be sufficiently covered, by other larger neighbourhoods (resulting from measurements), to ensure that one could describe it as 'locally compact', without leaving some ambiguity of, or uncertainty in, the physical interpretation of the notion. Therefore the equivalence of the definitions holds only when one considers the inner regions of sufficiently large subsets of parameter spaces of measurement values topologised by means of measurement processes.]

This equivalence has been discussed at length, as the following theorem will enter into the discussions of Section 3.3 on the topological properties of transformations in physics:

TH(10): Local compactness is invariant under interior mappings.]
Hocking & Young (1960), TH(2.51).

For the definition of an interior mapping see Section 3.2, DF(15).

3. *Topological Properties of Mappings*

3.1. *Introductory Remarks*

The reader will appreciate that the definition of a measurement process in DF(1 and 2) was designed so as to make it quite explicit that the numbers which result from any observation are but indirectly

† The meaning of this figurative description will become clearer to the reader in the light of the notions of equivalence and distinctness which are introduced in Section 5.3, DF(22).

related to the physical conditions examined. Since, moreover, the observational information is not quite precise, it is unlikely that any precise knowledge of the state of a physical condition can be elicited. One must therefore examine the possible different ways in which the imprecision in the domain and range of a measurement process may be tabulated. Neighbourhoods have already been seen to give a useful local description of the range, so they may be expected to be used again in the domain: in this way, then, it will be natural to introduce a topological classification of the mappings representing measurement processes and dynamical changes.

3.2. *Properties of Measurement Processes*

The discussion of Section 2 culminated in ASSN(3), which states that the range of a measurement process is Hausdorff and locally compact. In order to facilitate discussion, the following notations will be adopted:

CVN/NTN(1): The space of all physical conditions will be denoted by C.
CVN/NTN(2): A measurement process will be denoted as p.
CVN/NTN(3): The parameter space of measurement values will be denoted by F.
RMK(12): $\mathrm{dom}\,(p) \subseteq C$; $\mathrm{ran}\,(p) \subseteq F$.

In the demonstration of RMK(11), it was made apparent that it is not possible in physics to speak of points in $\mathrm{ran}(p)$, but only of minimal neighbourhoods. This has the consequence:
RMK(13): Even if there exist elements of C which may be mapped onto a point of convergence $\lim\limits_{\varepsilon\to 0} p_\varepsilon(c) \in F$, it is not possible to discover with complete certainty that such physical conditions exist. However, if such conditions do exist, which are precisely definable, then, as measurements are made more and more accurately, it should become progressively more possible to define unambiguous points of F. (The term 'unambiguous' must refer to some acceptable criterion.)
RMK(14): For the sake of strict accuracy we must notice that in the discussion of Section 2, p has been used in the sense that $p: C \to \mathcal{O}$, where $\mathcal{O} = \{O_i\}$ is a collection of sets covering $\mathrm{ran}\,(p)$ with a locally compact Hausdorff topology, whilst p was actually defined to be a mapping from C into F. There has been assumed to exist a natural map $\mathcal{O}_F: \mathcal{O} \to F$ which maps elements of $\mathcal{O}$ onto neighbourhoods of F in such a way that $\mathrm{ran}\,(p)$ has the topology described. Again, here it will be necessary to assume the existence of $\mathcal{O}_F$ in order to speak of p

as having topological properties. Indeed the whole possibility of giving any intuitively suggestive topological discussion depends upon the existence of $\mathcal{O}_F$!!

ASSN(4): There exists a natural map $\mathcal{O}_F$ from the set of measurements $\mathcal{O} = \{O_i\}$ (resulting from a measurement p) into a space F, such that to each element O_i there corresponds a subset $F_{0_i} = \mathcal{O}_F(O_i)$, and the collection $\{F_{0_i}\}$ gives a locally compact Hausdorff topology to the domain of $\mathcal{O}_F p$.

Having made this clarification we may now proceed to discuss the topological properties of p, considered as a mapping of C into F identical to $\mathcal{O}_F p$. Under this assumption, RMK(13) means that we may assume:

ASSN(5): Measurement processes preserve limit points.

Considering an open neighbourhood of a condition $c \in C$, we know that it cannot be mapped by p onto a single point of F. Since $p(c)$ is at smallest an open neighbourhood of a point in F, then p must take open neighbourhoods of C into open neighbourhoods of F. It does not follow from this that p is a one-to-one mapping (see Section 2.2 concerning the n, l, m, j quantum numbers of a hydrogen atom). Since there are no direct ways of ascertaining whether a measurement process can reveal all the limit points of a set of physical conditions, or of whether a corresponding set of limit points in F exists—except by intuitional hypothesising on the basis of observational values—it is not possible to make any direct statement about how p maps closed sub-sets of c. We may therefore state:

ASSN(6): Measurement processes map open sets into open sets.

In order to clarify matters a little further on, the following definition is introduced.

DF(10): A measurement process is called a *simple measurement process* if it is one-to-one.

The continuity properties of measuring processes can be examined by study of the relevant definitions and theorems in the following way:

DF(11): Given two sets S, T, each with a topology, a transformation $f: S \to T$ is called *continuous* if for any subset $X \subset S$ and point $x \in \bar{X}$, then $f(x) \in \overline{f(X)}$.

DF(12): A *homeomorphism* of S onto T is a one-to-one transformation $f: S \to T$ which is onto, and such that a point x, $x \in X \subset S$ is a limit point of X if and only if $f(x)$ is a limit point of $f(X)$.

Here we have used the notion of 'onto', which is defined as follows:

DF(13): A transformation g between two sets A, B, $g: A \to B$, is said to be *onto* if to every element of B there corresponds an element of A.

TH(6): A necessary and sufficient condition that a transformation $f: S \to T$ be continuous is that for any open set $U \subset T, f^{-1}(U)$ is open in S. **]** Hocking & Young (1960), TH(1.6).

TH(7): A necessary and sufficient condition that a transformation $f: S \to T$ be continuous is that if $x \in S$ and $f(x) \in V \subset T$, then there exists an open set $U \subset S$ such that $x \in U$ and $f(U) \subset V$. **]** Hocking & Young (1960), TH(1.7).

In order to avoid the possibility of a query about the omission of consideration of TH(6), we may point out that the condition contained in it is not of much use, for although we may be able to choose an open set $U \subset F$ by means of the union of measurement neighbourhoods, we have no means of actually verifying that $p^{-1}(U)$ is not closed in C. The reason being that there is no way of detecting all the limit points of U; for if that were possible one might hope to apply ASSN(5) if p were simple and then check that $p^{-1}(U)$ contained all its limit points.

The condition of TH(7) is rather more useful. Let $c \in C$, then $p(c) \in O_{\varepsilon_i}(a_i) \subset F$, where a_i is some measurement value with error $\varepsilon_i > \varepsilon_0$, where ε_0 is the minimal error. Let $O_{\varepsilon_i}(a_i)$ correspond to V in TH(7). Now if ε_i is sufficiently large, it will be possible to find—admittedly by inference—a small neighbourhood U of c such that $p_{\varepsilon_0}(U) \subset V = O_{\varepsilon_i}(a_i)$. (It is, of course, possible to consider V as being constructed from a large number of neighbourhoods of almost minimal size—'almost' minimal in order to ensure overlapping of adjacent neighbourhoods is possible—but it is rather more suggestive to consider the approach, used here, of very inaccurate measurements defining large open sets. The rest of the paragraph shows why.) As long as $O_{\varepsilon_i}(a_i) = V$ is very much larger than a minimal neighbourhood, the condition given by TH(7) is useful. But as soon as V becomes as small as a minimal neighbourhood, the physicist has at his disposal no neighbourhood in F—resulting from any measurement he may carry out—which can be strictly included in the minimal one. That is to say, the condition can no longer be used: there are no more appropriate entities to substitute in the condition that are the direct result of observational procedures. Therefore one cannot assume that a measurement process is continuous, merely on the basis of experimental procedures. However, if the assumption is taken *a priori*, it must be expected that as the error in a measurement process is reduced further, physical conditions which formerly seemed to be precisely definable will appear to have more and more internal structure. Only if this happens, can the condition of TH(7) be satisfied.

Before stating the final conclusion on the nature of measurement processes, we need the following definitions:

DF(14): An *open transformation* between two sets S, T is one for which the image of every open subset of S is open in T.

DF(15): An *interior transformation* is a continuous open transformation.

ASSN/RMK(7): To summarise: the practical limitations of finite, non-zero errors in measurements make it possible to state one thing only with certainty, namely that a measurement process is an open mapping. If a measurement process be in fact interior, then every increase in precision of measurements will reveal more details of structure in physical conditions.

We may add the following conclusion about simple measurement processes:

TH(8): A necessary and sufficient condition that a one-to-one transformation $f{:}S \rightarrow T$ be a homeomorphism is that f be interior.] Hocking & Young (1960), TH(1.9).

PR(2): A simple measurement process will always reveal finer details of physical conditions when measurements are made more accurate.]

3.3. *Topology of the Space of Physical Conditions*

Now that something of the nature of measurement processes is known, two invariance theorems about transformations may be coupled with our knowledge of the topology of F in order to obtain some idea of the topology of C. In particular we shall need the following two theorems, of which the second was mentioned at the end of Section 2.7.

TH(9): Compactness is invariant under continuous transformations.] Hocking & Young (1960), TH(1.24).

TH(10): Local compactness is invariant under interior transformations.] Hocking & Young (1960), TH(2.51).

Clearly the second theorem follows from the first. It is already known from the previous section that although a measurement process may not be strictly considered to be a continuous transformation between pairs of sets (of which those in F are smaller than a minimal neighbourhood), it is possible to note something of continuity on the coarser scale that ignores sets smaller than minimal neighbourhoods in F. As such sets are ignored in (observational) practice, we may think of measurement processes as continuous.† Therefore since the subspace of F in which measurement values appear is locally compact and Hausdorff, we may assume the same for G. But it is clear that this form of argument is not precise in any satisfactory way. These considerations may be summarised as follows:

ASSN(8): The region of the space of physical conditions that is accessible by a measurement process has a locally compact, Hausdorff topology.

3.4. *Dynamical Processes*

Measurement processes reveal no information about the ways in which changes between physical conditions occur, other than that different sets of measurement values specifying observed physical conditions are ordered into a sequence corresponding to the order in which the observations are made. The theoretical physicist is left to the task of constructing a mathematical formalism which will relate the observed measurement values to each other and accurately predict new measurement values. This can be characterised quite easily in terms of the notions we have developed so far. Let $F_1 \subset F$ be an initial set of measurement values characterising an initial condition, and let $F_2 \subset F$ be a final set. Then the transition which has been observed to have occurred is a transformation $\pi: F \to F$ with $\pi|F_1 : F_1 \to F_2$, where we have used the notation:

NTN(2): Given two sets S, T and a mapping $f: S \to T$, then if $A \subset S$, the *restriction mapping* of f such that A is mapped into T is denoted by $f|A$, or $f|_A$.

The corresponding physical process which gives rise to the transition $F_1 \to F_2$ is a mapping $\varpi: C \to C$, which also satisfies

$$\varpi|p^{-1}(F_1): p^{-1}(F_1) \to p^{-1}(F_2)$$

These notions can be expressed by the commutative diagrams:

$$\begin{tikzcd} C_1 \arrow[r, "\varpi|C_1"] \arrow[d, "p"'] & C_2 \arrow[d, "p"] \\ F_1 \arrow[r, "\pi|F_1"] & F_2 \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "\varpi"] \arrow[d, "p"'] & C \arrow[d, "p"] \\ F \arrow[r, "\pi"] & F \end{tikzcd}$$

† However, we shall be able to view this statement with more equanimity by referring the reader to the new notion of speiron-continuity, introduced in Section 5.3, DF(26).

The unusual notations π and ϖ (curly π) have been chosen for reasons which will become apparent in a later paper concerning fibre maps. We shall call the mapping ϖ a *dynamical process* for obvious reasons, but a more precise definition of the term will not be given until fuller considerations have been presented.

By reason of ASSN/RMK(7) we deduce that a dynamical process must be at least an open mapping. If, however, a measurement process does appear to be continuous, then the 'interior-ness' of it will imply that the associated dynamical process is also interior. This latter will result from the induced locally compact, Hausdorff topology of C, of ASSN(8). We may summarise:

PR(3): If a measurement process is open/interior, then the dynamical process relating physical conditions measured by the process is also open/interior, respectively.]

4. *Smirnov's Metrisation Theorems*

4.1. *Introductory Remarks*

It has become clear that the characterisation of physical conditions by means of elements of a space of parameters, that is a space of measurement values, leads us to a far less precise picture of the physical world than is currently used. Whereas nowadays physicists freely speak of points in space (of one kind or another) or use other 'sharply defined' notions, it has become clear that from a topological point of view it is impossible to assume the existence of such points; instead one has only neighbourhoods of a point, and any point can be postulated to exist only as the limit of a denumerable sequence of refined measurements, the limiting measurement having zero error.

RMK(15): It is therefore true to say that the physicist is in a far worse position than the topologist, for the topologist does actually have points in his topological spaces; the points are in fact the elements of a set to which a topology is given. The physicist, in contrast, has objects which within the limits of his perception only appear to be like those things a topologist calls neighbourhoods. But since the physicist has no such things as points, he can only assert that some measurement processes go so far to the limit of his perception of error that their measurement values seem to have properties in common with the topologist's points. Refinement of measuring techniques can always lead to the discovery of sub-structure within his 'point', however.

Metrisation is usually considered as the defining of a function over the cartesian product of a space with itself, which maps the product into a number field, and which function satisfies certain well-known axioms, namely:

DF(16): Given a space S, a *metric* over S is a function $d(S,S)$, defined on $S \times S$, with values in a number field such that the following conditions hold:

$$\begin{aligned} &AX/CDN(1): \quad d(x,y)=0 \Leftrightarrow x=y, \qquad \forall\, x,y \in S \\ &AX/CDN(2): \quad d(x,y)=d(y,x), \qquad \forall\, x,y \in S; \\ &AX/CDN(3): \quad d(x,z) \leqslant d(x,y)+d(y,z), \qquad \forall\, x,y,z \in S \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\textit{triangle inequality}). \end{aligned}$$

This notion relies upon that of the point, and so is an idealisation of any apparent metric properties of parameter spaces of measurement values. This observation has the following very important consequence:

RMK(16): Although a measurement process may donate a topology which is locally metrisable, it is not at all certain that it is globally metrisable. It is in fact known that a locally metrisable Hausdorff space is not metrisable except under a very particular condition [see TH(13)].

Therefore it is quite important to examine how closely the topological interpretation of physical limitations in measurement may restrict the validity of the notion of metrisation in physics. For instance, although we are familiar with metric properties of three-dimensional space in the macroscopic scale, we must seriously ask if the notion may still be valid at extreme distances.

4.2. *Paracompactness*

The most essential notion for our discussion of metrisability is the notion of paracompactness. It is very closely connected with the problem of reducing errors in measurement processes. First of all we need some preliminary notions.

NTN(3): If $\{V_\beta\}$ is a *refinement* of a covering $\{U_\alpha\}$, we write $\{U_\alpha\} < \{V_\beta\}$.

RMK(17): This quite clearly can be interpreted physically, by noticing that if ε_j is the error associated with a measurement process, and ε_i, $\varepsilon_i < \varepsilon_j$, is the error associated with a more accurate measurement process of the same kind, then if $\mathcal{O}_{\varepsilon_j} = \{O_{(\varepsilon_j)\alpha}\}$ and $\mathcal{O}_{\varepsilon_i} = \{O_{(\varepsilon_i)\beta}\}$ are the respective coverings of F we write $\mathcal{O}_{\varepsilon_j} < \mathcal{O}_{\varepsilon_i}$. For, quite clearly, the topology donated by $\mathcal{O}_{\varepsilon_i}$ is a refinement of that given by $\mathcal{O}_{\varepsilon_j}$.

DF(17): A covering $\{U_\alpha\}$ of a space S is called a *locally finite covering* if for each point $x \in S$ there exists an open set in S which contains x and intersects no more than a finite number of elements of $\{U_\alpha\}$.

RMK(18): By taking a covering $\mathcal{O}_\varepsilon = \{O_{(\varepsilon)}\}$ as being the collection of all subsets of F determined by a measurement process† any region of F (that is covered at all) can only be covered by a finite number of elements of $\mathcal{O}_\varepsilon$, owing to the impossibility of carrying out an infinite number of measurements in a finite time. Therefore we may say that a measurement process endows a locally finite covering on F.

DF(18): A space S is called *paracompact* if every covering of S possesses an open, locally finite refinement.

RMK(19): Since such parts of F as are covered by the neighbourhoods endowed by a measurement process have a definite, non-zero, minimum size, it is quite clear that no physical space can ever be discovered to be paracompact—for it is impossible for a finite number of observers to make an infinite number of observations in a finite time. This simple and rather naïve, point enables our conclusion to be asserted forcefully: for if one constructs a theory dependent upon a certain hypothesis, some way of testing the hypothesis must eventually be found if one ever wishes to ascertain the validity of the theory; paracompactness is essentially untestable. Therefore the following assumption is made explicit:

ASSN(9): No space of measurement values may be assumed to be paracompact.

4.3. *Lebesgue Numbers*

Here we introduce a notion parallel to that of minimal neighbourhoods of a covering of F, and that is used in mathematics, namely the Lebesgue number of a covering. The following theorem can be proven:

TH(11): Let M be a compact metric space, and let $\mathcal{U} = \{U_i\}$ be a finite open covering of M. Then there exists a positive number $d(\mathcal{U})$ such that each subset of M of diameter less than $d(\mathcal{U})$ is contained in at least one element of $\mathcal{U}$.] Hocking & Young (1960) TH(1.32)

CVN(4): The number $d(\mathcal{U})$ is called the *Lebesgue number* of the covering $\mathcal{U}$.

The parallelism is that if a set of physical conditions $X \subset C$ (or one physical condition) is capable of being more precisely specified in F than a certain measurement process p_ε indicates, then one may legitimately write the strict relation $p(X) \subset O_\varepsilon$, where O_ε is a minimal neighbourhood of the covering which p_ε gives to F, and p is a measurement process of greater accuracy than p_ε. Consequently, when a

† This means that a union of elements of $\mathcal{O}_\varepsilon$ is not counted as an element of $\mathcal{O}_\varepsilon$, but as an open set of the topology of F defined by $\mathcal{O}_\varepsilon$.

metric is introduced into local regions of F, it will always be possible to state that the typical set $p(X)$ lies within an element of the covering of F that has diameter ε. Thus ε is the Lebesgue number of the covering of F given by p_ε. We may say, then:

CVN(5): It is said that a *measurement process* p_ε has *Lebesgue number* ε.†

4.4. *Metrisation Theorems*

The first general theorem of Smirnov (1951) states the following condition:

TH(12): In order that a topological space S be metrisable, it is necessary and sufficient that it be regular and that it possess a basis $\mathscr{U}$ which is the union of not more than a denumerable number of locally finite systems $\mathscr{U}_n$.] Smirnov (1951), TH(1).

A slight generalisation of a locally finite covering has been used, namely:

DF(19): A system of sets $\mathscr{U}$ of a given topological space is called a *locally finite system (of sets)* if each point of the space has a neighbourhood intersecting a finite number of sets of the system $\mathscr{U}$ only.

A locally finite system can easily be made into a locally finite covering by adding as a new element the whole space. Therefore we do not have to make any investigation of the physical difference between the two notions. However, Smirnov does make the following important comment:

PR(4): If a T_1-space has a basis which is the union of a finite number of locally-finite systems, then it consists of isolated points.] Smirnov (1951), RMK(2).

RMK(20): In Section 2.6 it was shown that F is a T_1-space, then that the local compactness property of ASSN(3), DF(7) is equivalent to the locally finite system property. It is also clear that no more than a finite number of locally finite systems of measurement neighbourhoods can be defined in F by experimentation—for, as has been previously stated, a finite number of people operating measurement devices cannot produce an infinite number of results in a finite time. Application of PR(4) leads us to the conclusion that F consists of isolated points. In contradiction to that, it is not possible to detect isolated points by means of a measurement process, as we have been very

† Here we have assumed that F is locally metrisable; see DF(20), RMK(21).

careful to remark. Therefore this approach to the study of the topological properties of physical spaces—and indirectly, physical conditions—has an inherent inadequacy, for there are not enough tools of analysis to examine the physical implications of PR(4). Nevertheless, let us continue, in the hope that a fairly complete analysis in this vein will bring to light, directly or indirectly, inadequacies which may be rectified at a later date.

Returning to TH(12), it is known from ASSN(2) that the regularity of F can never be asserted, therefore TH(12) indicates that it is not possible to assert that F is a metric space. Looking at the $AX(T_1, T_3)$, it is plain that one may be able to approximate F to a metric space, however. For by TH(3), COR(1), ASSN(3), it is possible to define a small compact set—not minimally small—which is closed, and cover by disjoint open sets such a set as well as a point not included in the set. Therefore in this coarser sense (compared with the use of minimal neighbourhoods), F can be considered as approximately regular. Nevertheless one still is confronted with the difficulty of PR(4), RMK(20) above.

A second theorem of Smirnov (1951), TH(2), involves the normality property in the metrisation condition. But again, normality is essentially incapable of being tested. Smirnov's third theorem does give, however, a most useful criterion which is also an extremely general condition for metrisation:

TH(13): In order that a locally-metrisable Hausdorff space be metrisable, it is necessary and sufficient that it be paracompact.] Smirnov (1951), TH(3).

Here use has been made of the following notion:

DF(20): A space is called *locally metrisable* if each point of it has a neighbourhood which is a metric space.

RMK(21): Because one usually assumes that a measuring device gives sets of measurement values (corresponding to physical conditions) that behave according to the conditions of DF(16), that is to say the instinctive ordering a physicist gives to his measurement values is that of the real line (or higher dimensional Euclidean space), we may assume that F is at least locally metrisable. By RMK(19), ASSN(9), paracompactness is a property of F—and hence of C, by implication—which cannot be *a priori* assumed. Therefore we come to the following conclusion, which appears quite remarkable:

PR(5): *Even though a measurement process may donate a locally metrisable topology to a parameter space of measurement values F, the entirety of F may not be metrisable.*]

5. *Planted Structures*

5.1. *Introductory Remarks*

The simple observation that a measuring process $p_\varepsilon: C \to F$ does not map 'points' (i.e. well-defined elements) of C into points of F, but can at best map an element of C into a neighbourhood of F no smaller than a certain size characterised by ε, has led to difficulties over continuity of dynamical mappings and dynamical transformations. The difficulty is that any structure in C which a more refined measurement process would map into a region of F smaller than an 'ε-neighbourhood', is not distinguishable. It would therefore be desirable to be able to set up a mathematical formalism in which part of the structure may be hidden (i.e. ignored), and at the same time is provided with a means of uncovering the hidden structure by a process akin to increasing the accuracy of measurement. For this purpose we propose a new species of mathematical structure, which we shall name *Planted Structures*,† based upon some general notions of C. Ehresmann (1953).

5.2. *Species of Mathematical Structures*

It has already been made clear, in Section 2.3, that a topological structure on a set E is defined by giving a set $\mathcal{O}$, of subsets of E, which satisfies the union and intersection axioms of topological spaces, the elements $O_i \varepsilon \mathcal{O}$ being called the *open* subsets of E.

The set E can be considered as being quite arbitrary. The law of formation of $\mathscr{P}(\mathscr{P}(E))$, starting from E, together with the union and intersection axioms of a topological space define a species of topological structure, namely the class of topological structures on an arbitrary set.

Because the smallest possible observable differences in measurement values are involved in discussing the topological properties of dynamical mapping and dynamical processes, we must consider the notion of *local* structure, which Ehresmann (1953) defines as:

DF(21): A *species of local structures* is a species of structures (λ) for which there is given a law of induction—that is to say a law which associates to every structure S of a species (λ) given on a set E, a set Φ of subsets of E, and which determines on every set $U \in \Phi$ a structure of species (λ) called the *structure induced by S on U*, the set U provided with this structure being called a

† An example of a Zeeman 'Tolerance Space'.

distinguished subspace of E—in such a way that the following conditions are satisfied:

CDN(1): Φ is the set of open sets of a topology on E. It will be said that *S is a local structure with respect to the topology Φ*.

CDN(2): The law of induction is canonical, that is to say if a one-to-one mapping f of E onto E' transports the structure S onto a structure S' defined in E', then the restriction of f to such a distinguished subspace of E is an isomorphism onto a distinguished subspace of E' provided with the structure S'.

CDN(3): (*Transitivity of induced structures*): If U is a distinguished subspace of E, then the distinguished subspaces of U are the distinguished subspaces of E contained in U.

AX/CDN(4): (*Patching axiom*): If E' is the union of a family of subsets E_i of which each is provided with a structure of species (λ) such that $E_i \cap E_j$ be a distinguished subspace of E_i and E_j, the structures induced on $E_i \cap E_j$ by those which are given on E_i and E_j being identical, there exists on E' a well-defined structure of species (λ) such that E_i be a distinguished subspace of E' provided with this structure. We shall say that the space E' provided with this structure is obtained by *patching of subspaces* E_i. The structures given on the E_i will be called *coherent* amongst themselves.

There are two well-studied forms of local structure, called *Fibrous Structure* and *Foliate Structure*.

Planted Structures can possess both fibrous and foliate structures simultaneously, as will be apparent from their definition.

5.3. *Planted Structures*

DF(22): Let $\mathscr{V}$ be a covering, of the space E, with Lebesgue number $d(\mathscr{V})$. Denote by $\mathfrak{B}_{L(-)}$ the elements of $\mathscr{P}(E)$ which have diameter less than $d(\mathscr{V})$; denote by $\mathfrak{B}_{L(0)}$ the elements of $\mathscr{P}(E)$ which have diameter equal to $d(\mathscr{V})$, and denote by $\mathfrak{B}_{L(+)}$ those elements of $\mathscr{P}(E)$ which have diameter greater than $d(\mathscr{V})$. We now select certain subsets of E for use in defining the $\mathscr{V}$-planted structure $E_{\mathscr{V}}$, which satisfy the following conditions:

CDN(1): Any subset $U^{(0)} \subset E$ which belongs to $\mathfrak{B}_{L(0)}$ is called a *pip* of $E_{\mathscr{V}}$.

For any two subsets U_i and U_j of E, we say:

CDN(2): U_i is *$\mathscr{V}$-identical* to U_j if $[U_i - (U_i \cap U_j)] \in \mathfrak{B}_{L(-)}$.

CDN(3): U_i is *nominally $\mathscr{V}$-distinct* from U_j if

$$[U_i - (U_i \cap U_j)] \in \mathfrak{B}_{L(0)}.$$

CDN(4): U_i is *strictly $\mathscr{V}$-distinct* from U_j if

$$[U_i - (U_i \cap U_j)] \in \mathfrak{B}_{L(+)}$$

CDN(5): U_i and U_j are *$\mathscr{V}$-disjoint* if $(U_i \cap U_j) \in \mathfrak{B}_{L(-)}$.

These conditions may be strengthened as follows:

CDN((2)): U_i and U_j are said to be *$\mathscr{V}$-bi-identical* if each subset is $\mathscr{V}$-identical to the other.

CDN((3)): U_i and U_j are said to be *nominally $\mathscr{V}$-bidistinct* if each subset is nominally $\mathscr{V}$-distinct from the other.

CDN((4)): U_i and U_j are said to be *strictly $\mathscr{V}$-bidistinct* if each subset is strictly $\mathscr{V}$-distinct from the other.

The collection of all $\mathscr{V}$-distinct elements of $\mathscr{P}(E)$ is called the *generalised $\mathscr{V}$-planted structure on E*, and it will be denoted by $E_{\mathscr{V}}$.

RMK(22): A generalised $\mathscr{V}$-planted structure on a topological space E may therefore be built up as follows: (i) every set smaller than a $\mathfrak{B}_{L(0)}$-set is identified with the $E_{\mathscr{V}}$-pips in which it lies; (ii) every $\mathfrak{B}_{L(0)}$-set is an $E_{\mathscr{V}}$-pip, the smallest that can be considered in $E_{\mathscr{V}}$; (iii) any two sets of E which are $\mathscr{V}$-identical are represented by the same elements of $E_{\mathscr{V}}$; (iv) the first sets containing more than one $E_{\mathscr{V}}$-pip are those which are nominally $\mathscr{V}$-distinct from $E_{\mathscr{V}}$-pips, and they contain two $E_{\mathscr{V}}$-pips.

The multiple identification of subsets of E, which is used to create $E_{\mathscr{V}}$, produces a problem, in the simplest cases of (i) and (ii) above, concerning the way in which pips of $E_{\mathscr{V}}$ are packed together. The study of this problem is one in the field of homology theory, and in particular the general technique of Čech homology theory; it is initiated in Section 6.

Before we examine the first separation properties (cf. Section 2.6) of planted structures, we may consider two extreme cases of 'sowing' one topological space in another.

DF(23): Given two topological spaces E, F, and a covering $\mathscr{V}$ of F with Lebesgue number $d(\mathscr{V})$, we call a homomorphism $g_{\mathscr{V}}: E \to F_{\mathscr{V}}$ a *speiromorphism* of E into the $\mathscr{V}$-planted structure on F.

DF(24): In case (a), where $d(\mathscr{V}) = 0$, the speiromorphism $g_{\mathscr{V}}$ is called an *embedding* of E in F; in case (b), where $F_{\mathscr{V}}$ is a single $F_{\mathscr{V}}$-pip, then E is said to be *buried* in F by $g_{\mathscr{V}}$.

RMK/DF(23): Thus a speiromorphism is a generalisation of an embedding or immersion. If $\mathscr{U}$ is a covering of a topological space E with Lebesgue number $d(\mathscr{U})$, then a *speirojection* of E is a speiromorphism $g_{\mathscr{U}}: E \to E_{\mathscr{U}}$.

DF(25): Given two topological spaces E, F, and a covering $\mathscr{V}$ of F with Lebesgue number $d(\mathscr{V})$, we call a homomorphism $h_{\mathscr{V}}: F_{\mathscr{V}} \to E$ an *ekthamnomorphism* of the $\mathscr{V}$-planted structure on F into E. If given a speiromorphism $g_{\mathscr{V}}: E \to F_{\mathscr{V}}$, there exists an ekthamnomorphism $h_{\mathscr{V}}: F_{\mathscr{V}} \to E$ and such that $h_{\mathscr{V}} g_{\mathscr{V}} = 1_E$ and $g_{\mathscr{V}} h_{\mathscr{V}} = 1_F$, the identity mappings on E, F, respectively, then $g_{\mathscr{V}}$ is called a *speireomorphism.*

RMK/DF(24): In an obvious way we call an ekthamnomorphism $h_{\mathscr{U}}: E_{\mathscr{U}} \to E$ an *ekthamnojection.*

We can now re-shape the separation axioms so that they may apply to a planted structure of a topological space; there is assumed given a topological space E with an open covering $\mathscr{U}$, of Lebesgue number $d(\mathscr{U})$:

$AX(\mathscr{U} - T_0)$: Given two pips $y_1, y_2 \in E_{\mathscr{U}}$, there exist two $\mathscr{U}$-distinct open sets in E such that at least one of y_1, y_2 belongs to an open set not containing the other.

$AX(\mathscr{U} - T_1)$: Given two pips of $E_{\mathscr{U}}$, each one lies in one of a pair of open sets of E not containing the other pip, the pair of sets being at least $\mathscr{U}$-bidistinct.

$AX(\mathscr{U} - T_2)$: (Speiron-Hausdorff axiom): Given two pips of $E_{\mathscr{U}}$ there exist two $\mathscr{U}$-disjoint subsets each containing just one pip.

Using the necessary and sufficient condition for the continuity of a mapping given in TH(7), we may analogously define the notion of speiron-continuity of a speiromorphism in the following way:

DF(26): Given two topological spaces E, F, the latter possessing a planted structure $F_{\mathscr{V}}$, and a speiromorphism $g_{\mathscr{V}}: E \to F_{\mathscr{V}}$, then $g_{\mathscr{V}}$ is said to be *speiron-continuous* if for a point $x \in E$ and the pip $g_{\mathscr{V}}(x) \in F$ there exist two subsets $U \subset E$ and $V \subset F$ such that $x \in U$, $g_{\mathscr{V}}(x) \subset V$ and V is $\mathscr{V}$-distinct from $g_{\mathscr{V}}(U)$.

RMK(25): It is immediately noticed that a burying speiromorphism —or, more grotesquely, a 'burial'—is not speiron-continuous; however, an embedding speiromorphism is continuous in the ordinary sense. A homomorphism $g: E \to F$ has associated with it a homomorphism $g_{(*)}: E_{\mathscr{U}} \to F_{\mathscr{V}}$, which is also called a *speiromorphism.* For let $e_{\mathscr{U}}$ and $f_{\mathscr{V}}$ be the speirojections for E and F into $E_{\mathscr{U}}$ and $F_{\mathscr{V}}$ respectively, then $g_{(*)}$ is such that there holds $f_{\mathscr{V}} g = g_{(*)} e_{\mathscr{U}}$; that is to say we have the commutative diagram:

$$\begin{tikzcd} E \arrow[r, "g"] \arrow[d, "e_{\mathscr{U}}"'] & F \arrow[d, "f_{\mathscr{V}}"] \\ E_{\mathscr{U}} \arrow[r, "g_{(*)}"'] & F_{\mathscr{V}} \end{tikzcd}$$

The notion of *speiron-continuity* may also be similarly adapted: if x is a pip of E and $g_{(*)}$ is speiron-continuous, there exist two open sets $U \subset E$ and $V \subset F$ such that $x \in U$, $g_{(*)}e_{\mathscr{U}}(x) \subset g_{(*)}(U)$, $g_{(*)}(U) \subset V$ and that $g_{(*)}(U)$ is $\mathscr{V}$-distinct from V.

Referring back to Section 3 it is apparent that measurement processes fit into this structure, in particular:

PR(6): A measurement process $p_{\varepsilon}: C \to F$ is a speiromorphism $C \to F_{\mathscr{O}_{\varepsilon}}$.]

5.4. *Pip Set Topology of Planted Structures*

In Section 2 we considered aspects of topological spaces which are commonly known under the general title of 'Point Set Topology'. There is therefore the necessity of considering the speiron-analogues of various notions if we are to develop the use of speiromorphisms for representing measurement processes. A little has already been done by stating the axioms $AX(\mathscr{V} - T_0, T_1, T_2)$ and defining speiron-continuity (in what is, so far, a purely intuitive way), but we must go further, using, in the process, the distinguished subsets of planted structures, namely those that are distinct from pips. In the following let us again assume that E, F are topological spaces with coverings $\mathscr{U}$, $\mathscr{V}$, respectively, each covering having a Lebesgue number $d(\mathscr{U})$, $d(\mathscr{V})$ respectively. $E_{\mathscr{U}}$, $F_{\mathscr{V}}$ will be the respective planted structures resulting from application of the respective speirojections $e_{\mathscr{U}}$, $f_{\mathscr{V}}$.

First of all we need the notion of (distinguished)† open sets:

DF/NTN(27): A system of distinguished sets $\mathscr{U}^{(E)} = \{U_i^{(E)}\}$ is called a *covering* of $E_{\mathscr{U}}$ if $E_{\mathscr{U}} \subset \bigcup_i U_i^{(E)}$, and it is called an *open covering* of $E_{\mathscr{U}}$ if it satisfies:

CDN(1): The union of an arbitrary number of open† sets is an open set.

† From now on we shall assume that all sets in $E_{\mathscr{U}}$, $F_{\mathscr{V}}$ which are mentioned are in fact distinguished sets.

CDN(2): The intersection of a finite number of open sets is an open set.

There next follows the need for an analogue of limit points:

DF(28): If a pip $x \in E$ is a *limit pip of* $E_{\mathscr{U}}$, then every (distinguished) subset of E containing x also contains another distinct $E_{\mathscr{U}}$-pip.

RMK(26): It is immediately obvious that the usual notion of limit point cannot be taken straight into the field of planted structures, because there is no analogue of a sequence of progressively smaller sets. Therefore in a planted structure $E_{\mathscr{U}}$ an ordinary point of E has no special distinction from a limit point. The definition of a 'limit pip' given above is quite a different notion, for it defines what one might call 'pips at infinity' of $E_{\mathscr{U}}$, that is to say pips which are indefinitely remote from an arbitrary pip. Only when $d(\mathscr{U}) = 0$ does the notion of limit pip correspond to that of limit point, that is to say the correspondence does not hold unless $e_{\mathscr{U}}$ is an endomorphism of E.

From the notion of speiron-continuity of a speiromorphism $E_{\mathscr{U}} \to F_{\mathscr{V}}$ as enlarged in RMK(25), and from DF(28), it is easy to verify a result familiar in the limit of the speiromorphism being a map between two ordinary topological spaces:

PR(7): Limit pips are preserved under (speiron-) continuous speiromorphisms.]

An analogue of open maps may also be defined:

DF(29): An *open speiromorphism* transforms (distinguished) open sets into (distinguished) open sets.

Let us now look at the analogues of compactness and local compactness. We make the obvious choice of terminology.

DF(30): A planted structure $E_{\mathscr{U}}$ is said to be (*speiron-*) *compact* if it can be covered by a finite number of (distinguished) open sets.

PR(8): If $E_{\mathscr{U}}$ has at least one limit pip it is non-speiron-compact.]

DF(31): A planted structure is said to be *locally* (*speiron-*) *compact at a pip* x, if there is a speiron-compact (distinguished) open set containing x.

DF(32): A planted structure is said to be *locally* (*speiron-*) *compact* if it is locally speiron-compact at every pip.

From these definitions one immediately sees that:

PR(9): A planted structure is locally speiron-compact everywhere except at a limit pip.]

PR(10): A compact planted structure is locally compact.]

PR(11): Speiron-compactness and local speiron-compactness are invariant under speiromorphisms.]

PR(12): (Topological) compactness and local compactness are invariant under speirojections and speiromorphisms.]

6. *Čech Homology of Planted Structures*

6.1. *Introductory Remarks*

Following Section 5.3, RMK(22), we shall now examine a little of the homology properties of planted structures by Čech's techniques. The necessity for this arises from what can be called the 'packing' properties of planted structures—there will be many sets of a space E which are $\mathscr{U}$-identical to a pip $e_{\mathscr{U}}$ of the $\mathscr{U}$-planted structure $E_{\mathscr{U}}$, and many of the sets of E just $\mathscr{U}$-distinct from $e_{\mathscr{U}}$ will not be $\mathscr{U}$-distinct amongst themselves. This leads immediately to a multiple $\mathscr{U}$-identification of distinct sets (in E) about the pip $e_{\mathscr{U}}$, that is to say the sets of E which are just $\mathscr{U}$-distinct from $e_{\mathscr{U}} \in E_{\mathscr{U}}$ give rise to a certain number of pips which are packed around $e_{\mathscr{U}}$. Clearly it is a complicated situation to visualise; but since homology theory does describe something of the way in which bits and pieces of a space 'pack' together, it is natural to study the way in which the homology properties of planted structures might be discussed. It will become apparent that Čech theory is the most natural choice of technique. The treatment here follows Chapter 8 of Hocking & Young (1960), and a simple knowledge of chain groups, boundary operators and homology groups is assumed.

6.2. *Nerve of a Covering*

DF(33): Let E be a topological space, and $\Sigma(E)$ be the set of all open coverings $\mathscr{U}, \mathscr{V}, \ldots,$ of E, their Lebesgue numbers being denoted by $d(\mathscr{U}), d(\mathscr{V}), \ldots,$ respectively. In conventional Čech homology theory a covering $\mathscr{U}$ is considered as a simplicial complex by defining the elements of $\mathscr{U}$ as the *vertices*, and the sub-collections of distinct elements with non-vanishing intersections as the *simplexes*. The simplicial complex so constructed is called the *nerve of the covering* $\mathscr{U}$.

NTN(4): The nerve of a covering $\mathscr{U}$ is denoted as $\mathscr{N}(\mathscr{U})$. The adjustment of these notions to the planted structures is straightforward.

DF(34): Let E be a topological space and let $\Sigma(E)$ be the set of all open coverings $\mathscr{U}, \mathscr{V}, \ldots,$ of E; furthermore, let $\Sigma^{(E)}$ denote the set of all coverings $\mathscr{U}^{(E)}, \mathscr{V}^{(E)}, \ldots,$ of the respective planted structures $E_{\mathscr{U}}, E_{\mathscr{V}}, \ldots.$ A *vertex* of a covering $\mathscr{U}^{(E)}$ is taken as a

$\mathscr{U}$-distinguished set of $E_{\mathscr{U}}$, and a *simplex* as a sub-collection of mutually $\mathscr{U}$-distinct elements of $\mathscr{U}^{(E)}$ the intersection of which is a $\mathscr{U}$-distinguished set† (i.e. an element of $\mathscr{U}^{(E)}$) and is $\mathscr{U}$-distinct from each of the vertices. The resulting simplicial complex $\mathscr{N}(\mathscr{U}^{(E)})$ is called the *nerve of the covering* $\mathscr{U}^{(E)}$.

6.3. *Chain-Homotopic Projections*

In order to understand more clearly the complications existing in the Čech homology of planted structures, we must briefly examine a few of the preliminaries of the ordinary Čech homology of a topological space.

RMK(27): A partial ordering may be introduced into the set $\Sigma(E)$ by refinement. Also a covering $\mathscr{U} \cap \mathscr{V}$ of E may be defined as consisting of all non-empty intersections $U \cap V$ of $U \in \mathscr{U}$ and $V \in \mathscr{V}$. One easily sees that $\mathscr{U} \cap \mathscr{V} > \mathscr{U}$ and that $\mathscr{U} \cap \mathscr{V} > \mathscr{V}$. And hence we may consider $\Sigma(E)$ as a directed set under refinement.

RMK(28): There exists a simplicial mapping, called a *projection*, of a finer covering $\mathscr{V}$ into a coarser covering $\mathscr{U}$; we shall denote it as $\sigma_{\mathscr{U}\mathscr{V}}: \mathscr{V} \to \mathscr{U}$, $\mathscr{U}$, $\mathscr{V} \in \Sigma(E)$. It is defined by taking $\sigma_{\mathscr{U}\mathscr{V}}(V) = U$, $U \in \mathscr{U}$, $V \in \mathscr{V}$, where U is any fixed element of $\mathscr{U}$ such that $V \subset U$: this condition means that there can be many projections $\sigma_{\mathscr{U}\mathscr{V}}: \mathscr{V} \to \mathscr{U}$. Possible ambiguity owing to this multiplicity of projections is removed by the theorem:

TH(14): If $\mathscr{V} > \mathscr{U}$ in $\Sigma(E)$, then any two projections σ_1, σ_2 such that $\mathscr{V} \to \mathscr{U}$, are chain homotopic.] Hocking & Young (1960), TH(8.2.).

RMK(29): This result merely means that σ_1, σ_2 induce the same homomorphism σ_* between the groups $H_p(\mathscr{N}(\mathscr{V}), G)$ and $H_p(\mathscr{N}(\mathscr{U}), G)$, where G is the coefficient group.‡ Also we may state that if $\mathscr{W} > \mathscr{V} > \mathscr{U}$, then $\sigma_{\mathscr{U}\mathscr{V}*}\sigma_{\mathscr{V}\mathscr{W}*} = \sigma_{\mathscr{U}\mathscr{W}*}$. It is not necessary to recount any more in order to proceed with the qualitative discussion of the physical relevance of the application of these notions to planted structures.

RMK(30): Suppose that $\mathscr{V}^{(E)} > \mathscr{U}^{(E)}$ for two elements of $\Sigma^{(E)}$, then one expects from TH(14) that there will exist a chain homotopy $\sigma^{(E)}_{\mathscr{U}\mathscr{V}*}$: $H_p(\mathscr{N}(\mathscr{V}^{(E)}), G) \cong H_p(\mathscr{N}(\mathscr{V}^{(E)}), G)$ that is the same for all p.

† That is to say, at least as big as an $E_{\mathscr{U}}$-pip.

‡ It is sufficient to note that in general Čech homology theory it is known that there is some relationship between the combinations of the topological natures of $\mathscr{U}$, $\mathscr{V}$ and G which must be satisfied if the Čech homology groups are to exist.

However, this will not be true in general, owing to the dependence of $\mathscr{V}^{(E)}$ upon $d(\mathscr{V})$ and $\mathscr{U}^{(E)}$ upon $d(\mathscr{U})$. For instance, suppose that E is a metric space and is of minimum diameter ρ, and further suppose that the planted structure $E_{\mathscr{U}}$ buries E: then we know that $d(\mathscr{U})$ is such that U will not contain two sets U_1, $U_2 \in \mathfrak{U}_{L(0)}$ which are $\mathscr{U}$-distinct. In this case the only non-vanishing homology group is $H_0(\mathscr{N}(\mathscr{U}^{(E)}), G) = G$. Let $U_{i(0)}$, $i \in I_{(0)} \subset J_{\infty}^{(+)}$, be the coverings of E which are refinements of some others (the set $I_{(0)}$ indexing (i.e. directing) the refinements), the projections of which, $\sigma_{i(0)}$, are chain homotopic and induce the sequence of homomorphisms

$$\sigma_{(i-1,i)(0)*} : H_p(\mathscr{N}(\mathscr{U}_{i(0)}^{(E)}), G) \simeq H_p(\mathscr{N}(\mathscr{U}_{(i-1)(0)}^{(E)}), G)$$

The first homology group of a planted structure on E cannot exist until a refinement $\mathscr{V}$ of the covering of E is reached such that $E_{\mathscr{V}}$ consists of two $\mathscr{V}$-distinct pips. Let $\mathscr{V}_{i(1)}$, $i \in I_{(1)} \subset J_{\infty}^{(+)}$ be the set of coverings of E directed by refinement, as for $\mathscr{U}_{i(0)}$, such that $H_i(\mathscr{N}(\mathscr{V}_{i(1)}^{(E)}), G)$ may exist. Using an obvious extension of these ideas, we may state:

TH(15): If $\mathscr{W} > \mathscr{V} > \mathscr{U}$ in $\Sigma(E)$, then the induced homomorphisms $\sigma_{\mathscr{V}\mathscr{W}*}$ and $\sigma_{\mathscr{U}\mathscr{W}*}$ are, in general, different.]

CORR/RMK(TH(15)): In general, $\sigma_{\mathscr{V}\mathscr{W}*}$ and $\sigma_{\mathscr{U}\mathscr{W}*}$ are induced by different equivalence classes of projections.

RMK(31): Now if a topological space E is paracompact, the directed set of coverings $\Sigma(E)$ is denumerably infinite, and if $j \in J_{\infty}^{(+)}$, any sequence of refined coverings $\mathscr{U}_0$, $\mathscr{U}_1$, ..., $\mathscr{U}_j$, ... is such that the Lebesgue number $d(\mathscr{U}_j) \to 0$ as $j \to \infty$. Therefore, with this notation, we have the simple result:

PR(13): If E is paracompact and $\Sigma' \subset \Sigma(E)$ is an infinite, denumerable, directed set of refinements of coverings $\{\mathscr{U}_j\}$ of E, indexed by $J_{\infty}^{(+)}$, $j \in J_{\infty}^{(+)}$, then $E_{\mathscr{U}_j} \to E$ as $j \to \infty$.]

One might have expected PR(13) to be in the form $\lim_{j\to\infty} E_{\mathscr{U}_j} = E$, but it is by no means certain that, in all cases of a paracompact E, the equality will hold. Conditions for the existence of the limit are not studied here, but we shall give an example, EX(1) below, which illustrates the form of difficulty one may expect. Consider, for the moment that E is a foam rubber sponge, whose empty holes are all less than or equal to a certain volume V_E. Then any covering $\mathscr{U}$ with $d(\mathscr{U}) > V_E$ will make the sponge appear solid: let us denote the finest covering which does this by $\mathscr{U}_0$. Then refinements $\mathscr{U}_i$, constituting a directed set indexed by I, $(i \in I)$, will begin to reveal more and more of the structure, until all of even the smallest holes are revealed. But the situation can be made much more complicated: suppose now that E is a ship's biscuit with much the same structure, but that the biscuit is suddenly infested by microscopic quantum weevils. They will immediately start to burrow and tunnel (in the quantum sense, of course) and make the structure much finer. If they start to breed and produce baby quantum weevils, the structure will become even more riddled. Now it is easy to suppose that these quantum weevils will have quantum fleas upon their backs to bite'm, and they will have (quantum) littler fleas, and so add itchybitum. One can see that such a hypermultiplet structure can be summarised by the following definition:

DF(35): A topological space E is called *orthocompact* if for every pair of coverings $\mathscr{U}_j$, $\mathscr{U}_k$ with $\mathscr{U}_j > \mathscr{U}_k$ there exists a locally finite refinement $\mathscr{U}_i$ such that the induced homomorphisms

$$\sigma_{\mathscr{U}_j\mathscr{U}_i*}: H_p(\mathscr{N}(\mathscr{U}_i^{(E)}), G) \to H_p(\mathscr{N}(\mathscr{U}_j^{(E)}), G)$$

and

$$\sigma_{\mathscr{U}_k\mathscr{U}_i*}: H_p(\mathscr{N}(\mathscr{U}_i^{(E)}), G) \to H_p(\mathscr{N}(\mathscr{U}_k^{(E)}), G)$$

are not the same for all p.

RMK(32): Thus orthocompactness is a kind of wild speiron-compactness and wild paracompactness.

EX(1): An example of an object which is both paracompact and orthocompact is Alexander's horned sphere. It is paracompact because it is constructed by deforming a sphere and it is clearly orthocompact when one constructs a sequence of planted structures upon it that corresponds to the directed set of coverings of E.

RMK(33): Since open speiromorphisms have already been identified with measurement processes, it is clear that if the space of physical conditions is in fact orthocompact, then an infinite sequence of finite improvements in the error of measurement processes will reveal an infinite family of different structural properties of the space of (observed) physical conditions.

RMK(34): An advantage is now apparent in formulating physics in terms of planted structures; namely that a planted structure is not an idealisation, but is something that can be directly constructed from experiment. This is not true, for instance, of the supposition that three-dimensional (spatial) space is homeomorphic (in the embedding or immersion sense) to three-dimensional Euclidean space E^3, or even locally homeomorphic to E^3; such hypotheses, whilst very useful approximations in a multitude of cases, are not verifiable in the limit of zero experimental error.

7. *Conclusion*

7.1. *Conclusions*

We have seen a number of basic results, namely: (i) a measurement process defines a pre-sheaf over the space of physical conditions into the set of measurement value spaces; (ii) the strongest topology of a measurement value space that may be asserted is locally compact and Hausdorff; (iii) measurement processes are open mappings; (iv) dynamical processes are open mappings; (v) the regions of the space of physical conditions accessible by measurement processes have a locally compact Hausdorff topology; (vi) no space of measurement values may be assumed to be paracompact; (vii) even though a space of measurement values may be assumed to be locally metrisable, it is not necessarily metrisable throughout; (viii) a space of measurement values is a planted structure.

7.2. *Further Remarks*

Having shown that metrisability of measurement spaces, as opposed to local metrisability of them, may not be automatically assumed, one is forced to ask if it may be possible to express the notion of causality in a metric-free way. This problem is tackled in the next paper in this series,† and a solution is found.

There are two other problems suggested by this paper, and they are inter-related. The first is the calculation of the distance at which metrisation breaks down (to within a given error) from an arbitrary point in a locally metrisable non-paracompact space; the second is the construction of a global differential geometric formalism for planted structures.

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† On Causal Dynamics Without Metrisation: II. *International Journal of Theoretical Physics* (in press).

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